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A universal bound for a covering in regular posets and its application to pool testing *

Vladimir I. Levenshtein†

Keldysh Institute for Applied Mathematics, Russian Academy of Sciences, Miusskay Sq. 4, 125047 Moscow, Russia

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Let $V(n)$ be the set of all 2^n subsets of the set $N_n = \{1,2,...,n\}$ and $V_i(n) = \{x \in V(n) : |x| = i\}$. For a fixed $i = 1,...,n$, consider a covering operator $F : V_i(n) \to V(n)$ such that $x \subseteq F(x)$ for any $x \in V_i(n)$. Let $C = \{F(x) : x \in V_i(n)\}$. For any $1 \le T \le \binom{n}{i}$, consider the decreasing continuous function $g_i(T) = k + \frac{k+1}{i}(1-\alpha)$ where k and α are uniquely defined by the conditions $T\binom{k}{i} = \alpha\binom{n}{i}$, $k \in \{i,...,n\}$, and $1 - \frac{i}{k+1} < \alpha \le 1$. Using averaging and linear programing it is proved that

$$\frac{1}{\binom{n}{i}} \sum_{x \in V_i(n)} |F(x)| \ge g_i(|C|) \ge \frac{n}{\sqrt[i]{|C|}}$$

with the first inequality as an equality if and only if C is a Steiner $S(i, \{k, k+1\}, n)$ design with a specified distance distribution. A generalization of this result to the case of monotone left-regular n-posets is given. One of motivating applications is the problem of reconstructing an unknown binary vector $\mathbf{x}$ of length n using pool testing under the condition that the vectors $\mathbf{x}$ are

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†E-mail: leven@keldysh.ru

distributed with probabilities $p^{|x|}(1-p)^{n-|x|}$ where $x \in V(n)$ denotes the indices of the ones (active items) in $\mathbf{x}$. The bound above implies that the expected number of items which remain unresolved after application in parallel of arbitrary r pools is not less than

$$n \sum_{i=1}^{n} \binom{n}{i} p^i (1-p)^{n-i} 2^{-\frac{r}{i}} - np.$$

This improves upon an information theoretic bound for the minimum average number $E(n,p)$ of tests to reconstruct an unknown $\mathbf{x}$ using two-stage pool testing and allows determination of the asymptotic behavior of $E(n,p)$ up to a positive constant factor as $n \to \infty$ under some restrictions upon $p = p(n)$.